The $C_n^2$ Measurement Limits for a Horizontal Path with a Shack-Hartmann Wavefront Sensor

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1 Introduction

Several of our customers are using our Shack-Hartmann wavefront sensors to measure atmospheric parameters like $C_n^2$. As turbulence gets too great, the local wavefront slope can exceed the simple dynamic range of a Shack-Hartmann wavefront sensor. In this application note, we use Komogorov theory to establish the $C_n^2$ measurement limits using a Shack-Hartmann wavefront sensor without any slope dynamic range extension.

2 Approach and Results

I began by analyzing Fourier-generated Kolmogorov-spectrum phase screens to determine the ratio of the peak to RMS slope magnitude. I generated a set of Kolmogorov phase screens, calculated the gradient of the phase screens and established the phase slope magnitude from the calculated gradients. Then I calculated the RMS and peak amplitude for the 512x512 sample screens. Each screen had a nearly identical 7.5x ratio between the peak and RMS slope amplitudes, so I used this value for my analysis here. We also studied the PV to RMS ratio for the phase screens and found an average over 10 screens of 5.6±0.5.

For a horizontal path, Fried’s coherence length for a spherical wave is given by,$^1$

$$r_0 = 3.0 \left( C_n^2 L k^2 \right)^{\frac{3}{5}},$$

where $C_n^2$ is the index of refraction variance, r0 is Fried’s coherence length, L is the path length, k is the wave number. The 2D tilt variance for a single aperture is given by,

$$\alpha_{2D}^2 = 0.364 \left( \frac{D}{r_0} \right)^{\frac{5}{3}} \left( \frac{\lambda}{D} \right)^2,$$

where D is the aperture diameter and $\lambda$ is the wavelength. The RMS and peak tilt can be estimated by,

$$\alpha_{RMS} = \sqrt{\alpha_{2D}^2},$$

$$\alpha_{peak} = 7.5 \cdot \alpha_{RMS}.$$

Combining these we get a simplified equation for peak tilt from the fundamental parameters given by,$^1$

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The simple slope dynamic range for a Shack-Hartmann wavefront sensor (without any dynamic range extension) is given by,

\[
\theta_{\text{max,wfs}} = \frac{1}{M} \left( \frac{D}{2 f_{\text{sa}}} - \frac{1.22 \lambda}{D_{\text{sa}}} \right) = \frac{1}{M} \left( \frac{D}{2 f_{\text{sa}}} - \frac{1.22 \lambda}{D_{\text{sa}}} \right)
\]

where \( D_{\text{sa}} \) is the diameter of an individual sub-aperture, \( M \) is the magnification between the wavefront sensor and the output aperture of the telescope aperture, and \( f_{\text{sa}} \) is the focal length of the lens array. The diameter of the sub-aperture, \( D_{\text{sa}} \), can be scaled to the output aperture of the telescope with the optical system magnification, \( M \), by \( D = M \times D_{\text{sa}} \). Including this into the dynamic range equation, a relationship can be established for the maximum \( C_{n}^{2} \) as,

\[
C_{n}^{2 \text{max}} = \left( \frac{D^{1/6}}{11.38 \sqrt{L}} \right) \left( \frac{D}{2M f_{\text{sa}}} - \frac{1.22 \lambda}{D} \right)^{2}
\]

The approximate RMS slope noise floor for a Shack-Hartmann wavefront sensor at the output aperture of the telescope assuming the centroid accuracy is about 1/20th of a pixel is given by,

\[
\theta_{\text{min,wfs}} = \frac{1}{M} \frac{1}{20} \frac{d_{\text{pix}}}{f_{\text{sa}}},
\]

where \( d_{\text{pix}} \) is the diameter of a pixel. The minimum \( C_{n}^{2} \) can then be determined using this relationship as,

\[
C_{n}^{2 \text{min}} = 1.93 \times 10^{-5} \left( \frac{d_{\text{pix}}}{M} \right)^{2} \frac{D^{1/3}}{f_{\text{sa}}^{2} L},
\]

if the peak slope is set to the RMS slopes. If we set the RMS slopes to the RMS slope formula, we get a minimum \( C_{n}^{2} \) of,

\[
C_{n}^{2 \text{min}} = \frac{1}{900} \left( \frac{d_{\text{pix}}}{M} \right)^{2} \frac{1}{f_{\text{sa}}^{2} L}
\]

The ratio of the constant term in these formulas is 7.5²=57. Although this is a bit of a large range, the minimum detectable level is likely between these two values because of the spatial averaging that can occur making the real detectable limit about \( \sqrt{N_{\text{sa}}} \) lower than the individual sub-aperture RMS, where \( N_{\text{sa}} \) is the number of sub-apertures. Furthermore, the WFS is sensing the peak slopes, not the RMS wavefront slopes of the turbulence. Experimental calculations or careful modeling will be needed to determine the true nature of the minimum Cn2, but these formulas form a boundary estimate.

### 3 Summary

To examine these relationships, we executed a Matlab script to plot the maximum \( C_{n}^{2} \) for a system with varying magnification and different propagation lengths for a system with the parameters in Table 1. Figure 1
and Figure 2 show the maximum and minimum $C_n^2$ values respectively for varying magnification and different path lengths.

### Table 1: Example System Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{pix}$</td>
<td>7.4 microns</td>
</tr>
<tr>
<td>$f_{sa}$</td>
<td>6.7 mm</td>
</tr>
<tr>
<td>$D_{sa}$</td>
<td>150 microns</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>635nm</td>
</tr>
<tr>
<td>$D_{camera}$</td>
<td>3.6mm</td>
</tr>
</tbody>
</table>

![Figure 1: Example Maximum $C_n^2$ Values](image-url)
4  Acknowledgements

Special thanks to Jay Land for pointing out that the square root of the turbulent slope variance should be set to the RMS wavefront slopes instead of using the peak slope formula.