AN016

Working with Zernike Polynomials

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In this application note we are documenting some of the properties of Zernike polynomials and describing in more detail how they are used in the AOS software.

Zernike Decomposition Vector

The real Zernike polynomials are defined as [D. Malacara, Optical Shop Testing],

$$U_n^l = \begin{cases} \frac{1}{2}\left[Z_n^l + Z_n^{-l}\right] = R_n^l(\rho)\cos l\theta & \text{for } l \le 0, \\ \frac{1}{2i}\left[Z_n^l - Z_n^{-l}\right] = R_n^l(\rho)\sin l\theta & \text{for } l > 0, \end{cases}$$

where,

$$R_n^{n-2m}(\rho) = \sum_{s=0}^{m} (-1)^s \frac{(n-s)!}{s!(m-s)!\,(n-m-s)!}\,\rho^{n-2s}.$$

and n and l are the index parameters. In this software, the Zernike polynomials (aka Zernikes) are ordered in increasing order of n then l. The n index is effectively the radial power and l corresponds to the angular oscillation factor, which is valid from –n to +n in steps of 2. The first few terms in the n-l order are commonly referred to as x-axis tilt (henceforth x-axis will be x), y tilt, 90-degree astigmatism, focus, 45-degree astigmatism, x trefoil, x coma, y coma, y trefoil, x quadrafoil, 2nd-order x astigmatism, spherical aberration, 2nd-order y astigmatism, y quadrafoil, and x pentafoil. Since we are applying these polynomials to a Hartmann-type sensor, we are not using the first piston Zernike term.

Ordering

There are many different ways to order Zernike polynomials in the literature. The most common is the Noll ordering, but Wyant and Malacara also show representations. In general, these orderings go increasing radial power order (the n index), but differ in the order of the l term (the angular term). Table 1 summarizes the n and l terms for the different ordering schemes for the first 25 Zernike terms. From this ordering table, we can see some of the logic associated with the ordering. ***AOS uses an ordering we call n,l ordering in which we are sorting in increasing order the n and then the l indices of the Zernikes.*** Noll ordering sorts by increasing n then increasing l magnitude (absolute value) with positive l numbers before negative l numbers if n is even and negative l terms before positive l terms if n is odd. Malacara's ordering from his "Optical Shop Testing" book sorts by increasing n then decreasing l (or increasing m where m = (n-l)/2).

Table 1: Summary of Zernike Ordering Techniques

Index	AOS - NL Ordering			Noll			Malacara			Wyant		
	n	l	m	n	l	m	n	l	m	n	l	m
1	1	-1	1	1	-1	1	1	1	0	1	-1	1
2	1	1	0	1	1	0	1	-1	1	1	1	0
3	2	-2	2	2	0	1	2	2	0	2	0	1
4	2	0	1	2	2	0	2	0	1	2	-2	2
5	2	2	0	2	-2	2	2	-2	2	2	2	0
6	3	-3	3	3	1	1	3	3	0	3	-1	2
7	3	-1	2	3	-1	2	3	1	1	3	1	1
8	3	1	1	3	3	0	3	-1	2	4	0	2
9	3	3	0	3	-3	3	3	-3	3	3	-3	3
10	4	-4	4	4	0	2	4	4	0	3	3	0
11	4	-2	3	4	-2	3	4	2	1	4	-2	3
12	4	0	2	4	2	1	4	0	2	4	2	1
13	4	2	1	4	-4	4	4	-2	3	5	-1	3
14	4	4	0	4	4	0	4	-4	4	5	1	2
15	5	-5	5	5	-1	3	5	5	0	6	0	3
16	5	-3	4	5	1	2	5	3	1	4	-4	4
17	5	-1	3	5	-3	4	5	1	2	4	4	0
18	5	1	2	5	3	1	5	-1	3	5	-3	4
19	5	3	1	5	-5	5	5	-3	4	5	3	1
20	5	5	0	5	5	0	5	-5	5	6	-2	4
21	6	-6	6	6	0	3	6	6	0	6	2	2
22	6	-4	5	6	2	2	6	4	1	7	-1	4
23	6	-2	4	6	-2	4	6	2	2	7	1	3
24	6	0	3	6	4	1	6	0	3	8	0	4
25	6	2	2	6	-4	5	6	-2	4	5	-5	5

Analysis of Peak-to-Valley and RMS in Zernike Polynomials

Using the above formulation for Zernike polynomials (or the equivalent and easier to implement representation given in Hedser van Brug's SPIE paper "Efficient Cartesian representation of Zernike polynomials in computer memory"), we numerically studied the peak-to-valley (PV) amplitude and root-mean-squared (RMS) amplitude of the Zernikes over a unit radius circle using 1024 x 1024 grid of samples. Figure 1 shows the results of this analysis. The peak-to-valley amplitude is 2.0 for almost all the Zernikes with the exception of those with l=0 and n is a multiple of 4. In those cases, the peak-to-valley amplitude is $2^{-0.5}$ or approximately 1.41. The RMS amplitude has a general trend related to the radial index (n), but has anomalies when l=0 and n is a multiple of 2.

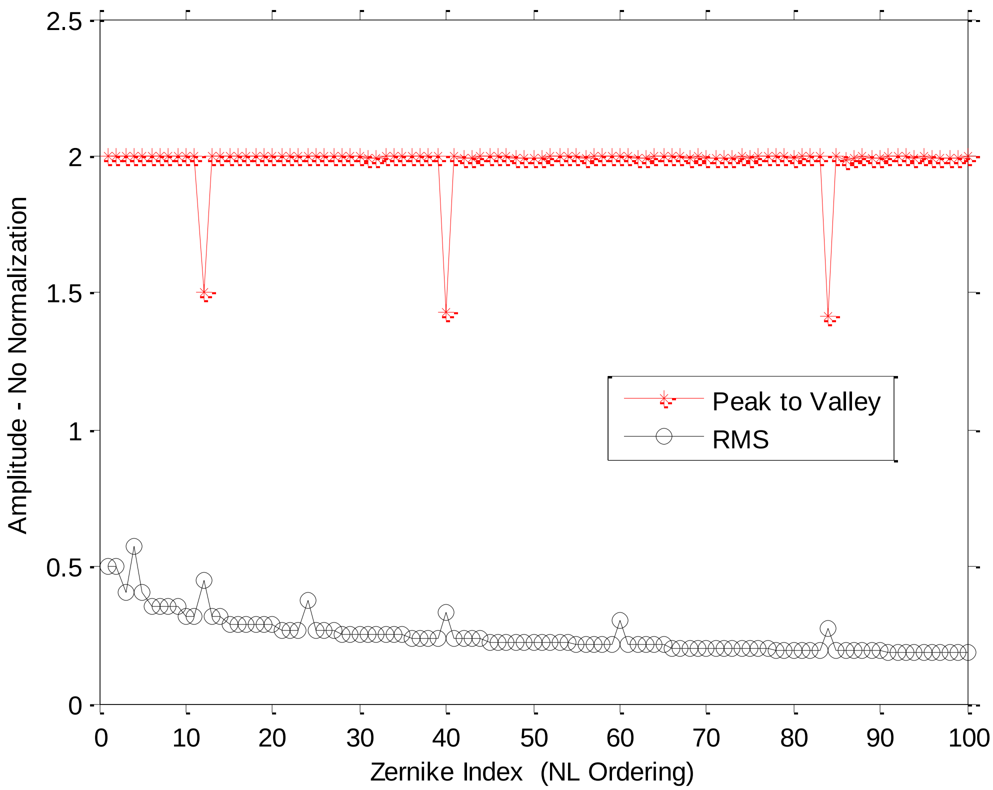
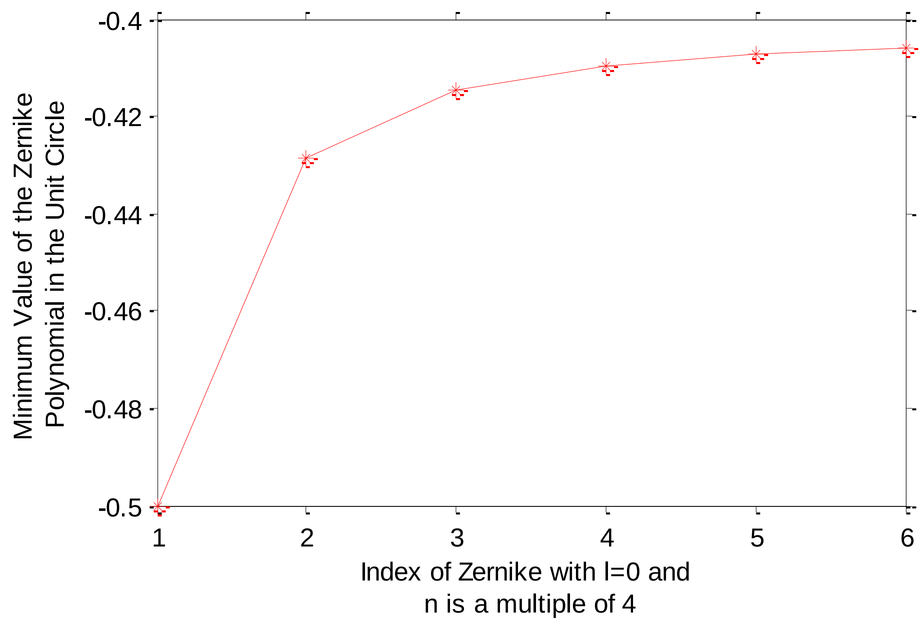


Figure 1 – Peak-to-Valley and RMS Amplitude for the first 100 Zernike Polynomials

Study of Anomalous Peak-to-Valley

Zernike terms where l=0 and n is a multiple of 4 show anomalous peak-to-valley amplitudes. These terms were analyzed numerically to see if there was indeed a simple discernable pattern in their peak-to-valley amplitude. Each of these Zernike terms showed a maximum of 1.0, but a minimum that varied from exactly -0.5 to a value near -0.4.



We analyzed the values numerically to try to discern an obvious pattern, but could not find a simple pattern. Suffice to say that the peak-to-valley amplitude of these Zernikes can be approximately reduced by a factor of 1/sqrt(2) and be accurate to within a few percent.

Table 2 – Analysis of the l=0,n =4*index Zernike Terms Minima

n	Minimum Value	Approximate Fraction	Fraction Value	Difference	PV Difference from √2
4	-0.5	-1/2	-0.5	0	0.085786
8	-0.428571429	-3/7	-0.42857	4.44089E-16	0.014358
12	-0.41475046	-90/217	-0.41475	3.9166E-06	0.000537
16	-0.409690446	-93/227	-0.40969	-1.18368E-06	-0.00452
20	-0.407276229	-347 / 852	-0.40728	-7.66205E-07	-0.00694
24	-0.405936584	-41 / 101	-0.40594	-4.01046E-06	-0.00828

PV to RMS Ratio Pattern

Analysis of the ratio of peak-to-valley to RMS illustrated a simple mathematical relationship relative to the radial power term (n) given by

$$\frac{\left(\frac{PV}{RMS}\right)^2}{(n+1)} = \begin{cases} 8, \text{ unless } l = 0 \\ 4, \text{ if } l = 0 \text{ and } m\%4 \neq 0 \text{ and } m\%2 = 0 \text{ (m is a multiple of 2, but not 4)} \\ 2, \text{ if } l = 0 \text{ and } m\%4 = 0 \text{ (m is a multiple of 4)} \end{cases}$$

Figure 2 shows this relationship for the first 100 Zernike terms.

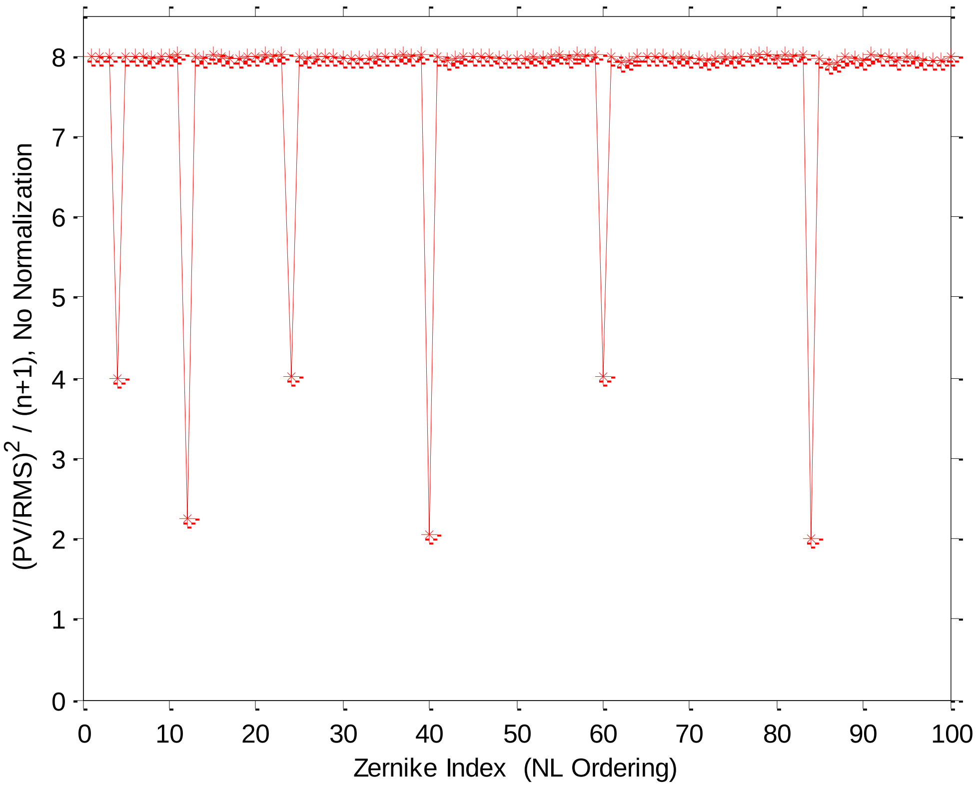


Figure 2 - Illustration of the simplification of the PV to RMS relationship with the n Zernike index.

Amplitude Scaling for Overlap Integral Orthonormalization

Zernikes are orthogonal about a unit circle, but their native overlap is not equal to one without proper scaling factors. In our software we have implemented the scaling factor such that the Zernikes self-overlap is unity. Using the polynomial definition described above, Malacara notes that the overlap of all Zernikes can be given by,

$$\int_0^1 \int_0^{2\pi} U_n^l \, U_{n'}^{l'} \rho \, d\rho \, d\theta = \frac{\pi}{2(n+1)} \delta_{nn'} \delta_{ll'} .$$

In the software, the overlapping Zernike is scaled by $\sqrt{\frac{2(n+1)}{\pi}}$. The vector will report the results of the decomposition with the traditional overlap integral normalization. The only exception we found to this was for the terms where n is a multiple of 4 and l is zero, in which case the normalization factor needs to be reduced by a factor of $\frac{1}{\sqrt{2}}$. Below is a table of the first few Zernikes and their associated numerical normalization factors.

Index	n	l	Name	Normalization Factor
1	1	-1	X Tilt	2
2	1	1	Y Tilt	2
3	2	-2	90-degree Astigmatism	2.4495
4	2	0	Focus	1.7321
5	2	2	45-degree Astigmatism	2.4495
6	3	-3	X Trefoil	2.8284
7	3	-1	X Coma	2.8284
8	3	1	Y Coma	2.8284
9	3	3	Y Trefoil	2.8284
10	4	-4	X Quadrafoil	3.1623
11	4	-2	X 2nd-order Astigmatism	3.1623
12	4	0	Spherical Aberration	2.2361
13	4	2	Y 2nd-order Astigmatism	3.1623
14	4	4	Y Quadrafoil	3.1623
15	5	-5	X Pentafoil	3.4641

Peak-to-Valley Zernike Normalization

The raw Zernike polynomials are scaled with the following code to create a near unity peak to valley:

```
x=0.5;
if (mod(n,4)==0 && l==0) x=1/sqrt(2); end;
if (n==4 && l==0) x=1/1.5; end;
if (n==8 && l==0) x=1/(1+3/7); end;
```

Overlap-Normalized Scaling

The raw Zernike polynomials are scaled with the following code to create a near unity overlap integral:

```
x=sqrt(2.0*(n+1.0)/pi);
if (mod(n,2)==0 && l==0); x=x./sqrt(2); end;
```

Overlap Integral Calculation for Zernike Decomposition

To start calculating the overlap integral (aka dot product) we multiplied each sample by its corresponding point in the overlap-normalized Zernike term over the unit-radius circle and then divided by the number of points. The code for this in Matlab can be implemented as "`coef(ii) = sum(sum(zideal.*ztest)) ./ cnt;`". This methodology does not take into account the area factor, so when we evaluated the self-overlap integral between the overlap normalized Zernike polynomials over the first 100 Zernikes and found that the result was $1/\pi$. To complete the overlap integral, we need to multiply by the area of the unit circle, which is π, and leaves unity.

Comparison of RMS to Overlap-Normalized Overlap Integral

We performed the overlap integral on a Zernike term that had been scaled to have a peak-to-valley amplitude of 1 micron with its corresponding overlap-normalized Zernike term without multiplying by the area (π). Figure 3 shows the results from this analysis. The resulting overlap coefficient was found to be equal to the RMS wavefront of the test Zernike divided by √π within the numerical noise. If the proper area factor is used, the RMS Zernike amplitude would be the overlap integral divided by √π instead of times √π.

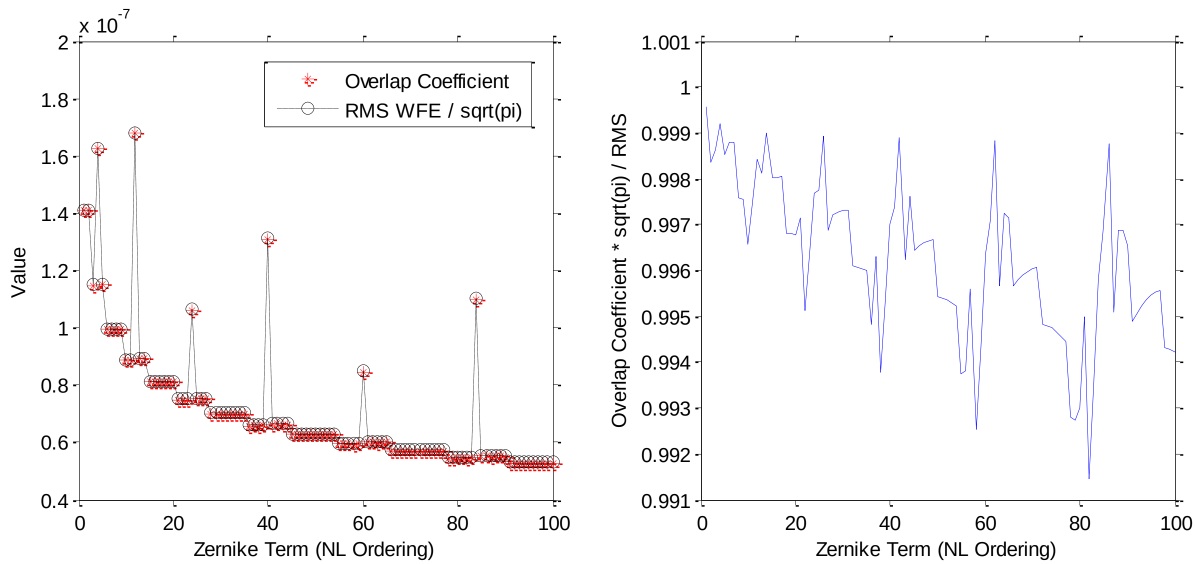


Figure 3 - Results of Overlap Integral between 1-micron PV amplitude Zernike and the Overlap-Normalized Zernike Term

Conclusions

Zernike polynomials are widely used in optics, but have different orderings and different amplitude scale factors. The AOS software uses the Zernike terms ordered by increasing n then increasing l (NL ordering). The software reports the overlap-normalized (orthonormalized) coefficients as the Zernike polynomial decomposition. The RMS amplitude of each Zernike from the decomposition is equal to the overlap integral coefficient times √π.

References

D. Malacara , Optical Shop Testing, Wiley-Interscience; 2 edition (January 1992).
Hedser van Brug, "Efficient Cartesian representation of Zernike polynomials in computer memory", Proc. SPIE Vol. 3190, p. 382-392.
http://www.optics.arizona.edu/jcwyant/zernikes/Zernikes.pdf (Applied Optics and Optical Engineering, Vol. XI., Chapter 1)